Journal of Engineering Mathematics **34:** 435–450, 1998.

Net flow of compressible viscous liquids induced by travelling waves in porous media

A. C. T. AARTS and G. OOMS
Shell International Exploration and Production B.V., Research and Technical Services, P.O. Box 60, 2280 AB Rijswijk, The Netherlands. e-mail: a.c.t.aarts@siep.shell.com

Received 19 May 1997; accepted in revised form 3 August 1998

Abstract. The influence of ultrasonic radiation on the flow of a liquid through a porous medium is analyzed. The analysis is based on a mechanism proposed by Ganiev *et al.* according to which ultrasonic radiation deforms the walls of the pores in the shape of travelling transversal waves. Like in peristaltic pumping, the travelling transversal wave induces a net flow of the liquid inside the pore. In this article, the wave amplitude is related to the power output of an acoustic source, while the wave speed is expressed in terms of the shear modulus of the porous medium. The viscosity as well as the compressibility of the liquid are taken into account. The Navier–Stokes equations for an axisymmetric cylindrical pore are solved by means of a perturbation analysis, in which the ratio of the wave amplitude to the radius of the pore is the small parameter. In the second-order approximation a net flow induced by the travelling wave is found. For various values of the compressibility of the liquid, the Reynolds number and the frequency of the wave, the net flow rate is calculated. The calculations disclose that the compressibility of the liquid has a strong influence on the net flow induced. Furthermore, by a comparison with the flow induced by the pressure gradient in an oil reservoir, the net flow induced by a travelling wave can not be neglected, although it is a second-order effect.

Keywords: acoustics, peristaltic transport, porous-media flow, travelling waves, well stimulation.

1. Introduction

Laboratory experiments have shown that ultrasonic radiation can considerably increase the rate of flow of a liquid through a porous medium. For instance, Chen [1] investigated the influence of ultrasonic radiation (at a frequency of 20 kHz) on the flow of water or oil through a stainless steel filter, on the flow of oil through porous sandstone samples, and on the flow of oil segments through a capillary. For the flow through sandstone samples he found that ultrasonic radiation increased the oil-flow rate by a factor of three. Although the ultrasonic energy heated the oil, and hence decreased its viscosity, he showed that only a part of the increase of the flow rate could have been caused by the viscosity decrease. Cherskiy *et al.* [2] measured the permeability of core samples saturated with fresh water under the influence of an acoustic wave field (at a frequency of 26·5 kHz). They measured a sharp increase of the permeability within a few seconds after the beginning of the ultrasonic treatment. After removal of the sound field the permeability decreased to the value before radiation. Duhon [3] studied the characteristics of oil displacement by water in sandstone samples in the presence of ultrasound (at frequencies of 1·0–5·5 MHz). He observed an additional amount of oil recovered of 6–15% and an increase of the flow rate.

The oil industry has shown great interest in the effects of ultrasound on the flow through a porous medium. The reason is that in production the oil flows due to a pressure gradient through a porous medium in the underground, the so-called reservoir, into a well. The effects on flow rates observed in experiments thus reveal that ultrasonic irradiation of the reservoir downhole in a well is capable of increasing oil production.

A theoretical explanation for the increase of the rate of flow of a liquid through a porous medium by ultrasonic radiation has been given by Ganiev *et al.* [4]. They propose that ultrasonic radiation generates travelling transversal waves at the pore walls in a porous medium. These waves induce in the liquid inside the pores internal waves of velocity and pressure with a nonuniform amplitude distribution along the radius of the pore. Ganiev *et al.* [4] consider a travelling transversal wave on the wall of a single pore (approximated by a cylindrical tube of small radius) which is filled with a compressible viscous liquid. After carrying out a perturbation analysis, in which the ratio of the wave amplitude to the radius of the pore is the small parameter, they found in the second-order approximation a net flow of the liquid. The paper of Ganiev *et al.* [4], however, is very concise and does not report in detail the analysis and the calculations. Ganiev *et al.* only formulate the basic equations and give some results. Furthermore, they do not analyse the influence of the acoustic power output and the material properties of the porous medium and the liquid.

The proposed mechanism of Ganiev *et al.* [4] is identical to the peristaltic transport mechanism. Peristaltic pumping is often used in medical instruments such as the heart-lung machine. It frequently occurs in the organs in the living body like the ureters, intestines and arterioles. In peristaltic pumping a travelling transversal wave generated along a flexible wall propels the liquid along a tube. In peristaltic pumping, however, usually the transport of incompressible liquids is described. Various papers have been written about peristaltic transport, see for instance Shapiro *et al.* [5], Yin and Fung [6], and Takabatake *et al.* [7]. In particular, Yin and Fung [6] studied the flow of an incompressible viscous liquid through a cylindrical tube with its wall deformed in the shape of a travelling transversal wave. After making their dimensionless, they found three relevant dimensionless parameters, *viz.* $\varepsilon = a/R$, $\alpha = 2\pi R/\lambda$, and the Reynolds number $\mathrm{Re} = Rc\rho/\mu$. Here, a is the amplitude of the wave, R is the radius of the tube, λ is the wavelength, and c is the wave speed, while ρ is the density and μ the viscosity of the liquid. Like Ganiev *et al.* [4], Yin and Fung [6] carry out a perturbation analysis with ε as small parameter. Later, Takabatake *et al.* [7] mention that the perturbation analysis of Yin and Fung [6] is only valid if the dimensionless parameters ε, α and Re satisfy $\varepsilon\alpha^2\,\mathrm{Re} \ll 1$.

In this article we elaborate the mechanism proposed by Ganiev *et al.* [4], in order to analyze the influence of ultrasonic radiation on flow through porous media. Although natural porous media are a random network of interacting capillaries of variable cross-section, we use a simplified representation in which the porous medium consists of a set of noninteracting, straight and circular capillaries in an elastic medium. We consider a flexible axisymmetric cylindrical pore of radius R, with its wall deformed by the application of ultrasound in the shape of a travelling transversal wave of constant amplitude a. The pore is filled with a compressible viscous liquid which is at rest in the absence of the travelling wave. We render our problem dimensionless and carry out a perturbation analysis for small ratios of $\varepsilon = a/R$. In the second-order approximation a net flow induced by the travelling wave is found. Actually, we extend the analysis of Yin and Fung [6] by taking the compressibility κ of the liquid into account. As a result we obtain, in addition to ε, α and Re, a fourth relevant dimensionless parameter, *viz.* $\chi = \kappa\rho_0 c^2$, where ρ_0 is the (constant) density at a reference pressure. Although the compressibility κ is small, the dimensionless parameter χ has a magnitude of order 1 for the flows considered. Calculations for various values of ε, α, Re and χ, disclose that the

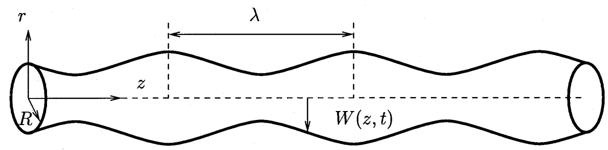

Figure 1. The cylindrical pore with its wall deformed in the shape of a travelling transversal wave.

compressibility of the liquid has a strong influence on the net flow induced. Hence, although the liquids considered are hard to compress, their compressibility may not be neglected.

Finally, we give an estimation of the dimensionless parameter ε in ultrasonic radiation of a reservoir downhole, and compare the net flow induced by a travelling wave to the flow induced by the pressure gradient in the reservoir. To that end, we express the wave amplitude a in terms of the power output P generated at the source, and we relate the wave speed c to the shear modulus G of the porous medium. By means of an example, in which we take $P = 10$ kW and $G = 0{\cdot}5 \times 10^9$ N/m^2, we show that ε has a magnitude of order 10^{-3} in the reservoir near the well (the so-called near-wellbore region), while the wave speed c is of order 10^3 m/s. Thus, a perturbation analysis with ε as a small parameter is permitted. With a frequency in the order of 10^4 Hz and a pore radius in the order of 10^{-4} m, the wavelength λ is in the order of 10^{-2} m, while the Reynolds number is in the order of 10^4. In the example we demonstrate that the net flow rate due to a travelling wave with small dimensionless amplitude is of the same order of magnitude as the Poiseuille flow rate due to the pressure gradient in the reservoir. Hence, although the net flow induced by travelling waves is an effect which is proportional to ε^2, this effect cannot be neglected in the flow through a reservoir.

2. Formulation of the problem

A flexible axisymmetric cylindrical pore of radius R is considered. The wall of the pore is deformed in the shape of a travelling transversal wave with constant amplitude a. Cylindrical coordinates (r, θ, z) are introduced with the z-axis along the centerline of the pore. With the wave travelling along the z-axis, the wall $r = W(z, t)$ of the pore takes the form

$$W(z, t) = R + a \cos\left(\frac{2\pi}{\lambda}(z - ct)\right), \tag{2.1}$$

where λ is the wavelength and c is the wave speed; see Figure 1.

A compressible viscous liquid is flowing through the pore. The flow is governed by the balance of mass

$$\frac{\partial\rho}{\partial t} + (\mathbf{v} \cdot \nabla\rho) + \rho(\nabla \cdot \mathbf{v}) = 0 \tag{2.2}$$

and the Navier–Stokes equation

$$-\nabla p + \mu\nabla^2\mathbf{v} + \frac{\mu}{3}\nabla(\nabla \cdot \mathbf{v}) = \rho\frac{\partial\mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v}, \tag{2.3}$$

which represents the balance of linear momentum under neglect of gravity for a so-called Stokes fluid. Here, ρ is the liquid density, p the pressure, $\mathbf{v}$ the liquid velocity and μ the viscosity of the liquid. The characteristic response of the liquid to a compression is described by the constitutive equation

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}p} = \kappa, \tag{2.4}$$

where κ is the compressibility of the liquid.

With the flow parameters independent of the azimuthal coordinate θ, the velocity takes the form

$$\mathbf{v} = v_r(r, z, t)\mathbf{e}_r + v_z(r, z, t)\mathbf{e}_z, \tag{2.5}$$

where $\mathbf{e}_r$ and $\mathbf{e}_z$ are unit vectors in the positive r- and z-direction, respectively. At the boundary $r = W(z, t)$ the liquid is subjected to the motion of the wall. The no-slip boundary condition at the wall thus requires

$$v_r(W, z, t) = \frac{\partial W}{\partial t}, \qquad v_z(W, z, t) = 0. \tag{2.6}$$

The solution of (2.4) for the density as function of the pressure is given by

$$\rho = \rho_0 \exp[\kappa(p - p_0)], \tag{2.7}$$

where ρ_0 is the (constant) density at the reference pressure p_0. In cylindrical coordinates, the balance of mass (2.2) reads

$$\frac{\partial \rho}{\partial t} + v_r\frac{\partial \rho}{\partial r} + v_z\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}\right) = 0, \tag{2.8}$$

while the Navier–Stokes equation (2.3) becomes

$$\begin{aligned} -\frac{\partial p}{\partial r} \;+\; \mu\left(\frac{\partial v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}\right) + \frac{\mu}{3}\frac{\partial}{\partial r}\left\{\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}\right\} \\ = \rho\frac{\partial v_r}{\partial t} + \rho\left(v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\right), \end{aligned} \tag{2.9}$$

$$\begin{aligned} -\frac{\partial p}{\partial z} \;+\; \mu\left(\frac{\partial v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial v_z}{\partial z^2}\right) + \frac{\mu}{3}\frac{\partial}{\partial z}\left\{\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}\right\} \\ = \rho\frac{\partial v_z}{\partial t} + \rho\left(v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right). \end{aligned}$$

Finally, the flow rate Q through the pore is given by

$$Q = Q(z, t) = 2\pi\int_0^{W(z,t)} v_z(r, z, t)r\,\mathrm{d}r. \tag{2.10}$$

The equations are made dimensionless by scaling length by R and time by R/c. Furthermore, we introduce the dimensionless variables $\tilde{W} = W/R$, $\tilde{\rho} = \rho/\rho_0$, $\tilde{v_r} = v_r/c$, $\tilde{v_z} = v_z/c$,

$\tilde{p} = p/\rho_0 c^2$, $\tilde{p}_0 = p_0/\rho_0 c^2$ and $\tilde{Q} = Q/cR^2$, and the four dimensionless parameters ε, α, Re and χ given by

$$\varepsilon = \frac{a}{R}, \quad \alpha = \frac{2\pi R}{\lambda}, \quad \mathrm{Re} = \frac{\rho_0 c R}{\mu}, \quad \chi = \kappa \rho_0 c^2. \tag{2.11}$$

Henceforth, we will omit the tildes without ambiguity. Then the Equations (2.9) turn into their dimensionless form, reading

$$\begin{aligned} -\frac{\partial p}{\partial r} &+ \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}\right) + \frac{1}{3\,\mathrm{Re}}\frac{\partial}{\partial r}\left\{\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}\right\} \\ &= \rho\frac{\partial v_r}{\partial t} + \rho\left(v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\right), \end{aligned} \tag{2.12}$$

$$\begin{aligned} -\frac{\partial p}{\partial z} &+ \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right) + \frac{1}{3\,\mathrm{Re}}\frac{\partial}{\partial z}\left\{\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}\right\} \\ &= \rho\frac{\partial v_z}{\partial t} + \rho\left(v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right), \end{aligned}$$

while Equation (2.7) turns into its dimensionless form, reading

$$\rho = \exp[\chi(p - p_0)]. \tag{2.13}$$

The balance of mass (2.8) in terms of ρ, v_r and v_z remains the same after scaling, and also holds for the dimensionless variables $\tilde{\rho}$, $\tilde{v_r}$ and $\tilde{v_z}$. The boundary conditions (2.6) for v_r and v_z at $r = W$ remain the same after scaling, and also hold for the dimensionless variables $\tilde{v_r}$ and $\tilde{v_z}$ at $\tilde{r} = \tilde{W}$, where W in its dimensionless form reads

$$W(z,t) = 1 + \eta(z,t), \qquad \eta(z,t) := \varepsilon \cos \alpha(z - t). \tag{2.14}$$

Thus, the boundary conditions in their dimensionless form read

$$v_r(1 + \eta(z,t), z, t) = \frac{\partial \eta(z,t)}{\partial t}, \qquad v_z(1 + \eta(z,t), z, t) = 0. \tag{2.15}$$

Relation (2.10) which defines the flow rate Q in terms of v_z and W remains the same after scaling, and also holds for the dimensionless variables $\tilde{Q}$, $\tilde{v_z}$ and $\tilde{W}$. The parameter Re is the Reynolds number. The number $\sqrt{\chi} = c\sqrt{\kappa\rho_0}$ represents the ratio of the wave speed c on the pore wall, and $c_0 := 1/\sqrt{\kappa\rho_0}$ is the speed of sound in the liquid. For the relatively small wave amplitudes considered here we have $\varepsilon \ll 1$.

3. Method of solution

To illustrate the nature of the solution we shall consider the important case of no flow in absence of the travelling wave. As a consequence, the pressure p is equal to the reference pressure p_0, if $\varepsilon = 0$. Therefore, we assume solutions for the pressure, velocity and density in the form

$$\begin{aligned} p &= p_0 + \varepsilon p_1(r,z,t) + \varepsilon^2 p_2(r,z,t) + \cdots, \\ v_r &= \varepsilon u_1(r,z,t) + \varepsilon^2 u_2(r,z,t) + \cdots, \\ v_z &= \varepsilon v_1(r,z,t) + \varepsilon^2 v_2(r,z,t) + \cdots, \\ \rho &= 1 + \varepsilon \rho_1(r,z,t) + \varepsilon^2 \rho_2(r,z,t) + \cdots. \end{aligned} \tag{3.1}$$

Notice that Yin and Fung [6] have included a zeroth-order constant pressure gradient $\mathrm{d}p_0/\mathrm{d}z$ in their solution for an incompressible liquid, so that the axial velocity v_z contains the zeroth-order term

$$V_0(r) = -\frac{\mathrm{Re}}{4}\frac{\mathrm{d}p_0}{\mathrm{d}z}(1-r^2). \tag{3.2}$$

The calculations in [6], however, were carried out for $\mathrm{d}p_0/\mathrm{d}z = 0$. Substituting the expansions (3.1) in (2.8), (2.12) and (2.13), and equating the coefficients of equal powers of ε on both sides of the equations, we obtain the following sequence of equations:

$$\begin{aligned} &-\frac{\partial p_1}{\partial r} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} - \frac{u_1}{r^2} + \frac{\partial^2 u_1}{\partial z^2}\right) \\ &\qquad + \frac{1}{3\,\mathrm{Re}}\frac{\partial}{\partial r}\left\{\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{\partial v_1}{\partial z}\right\} = \frac{\partial u_1}{\partial t}, \\ &-\frac{\partial p_1}{\partial z} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\frac{\partial v_1}{\partial r} + \frac{\partial^2 v_1}{\partial z^2}\right) \\ &\qquad + \frac{1}{3\,\mathrm{Re}}\frac{\partial}{\partial z}\left\{\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{\partial v_1}{\partial z}\right\} = \frac{\partial v_1}{\partial t}, \\ &\frac{\partial \rho_1}{\partial t} + \frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{\partial v_1}{\partial z} = 0, \qquad \rho_1 = \chi p_1; \end{aligned} \tag{3.3}$$

$$\begin{aligned} &-\frac{\partial p_2}{\partial r} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} - \frac{u_2}{r^2} + \frac{\partial^2 u_2}{\partial z^2}\right) + \frac{1}{3\,\mathrm{Re}}\frac{\partial}{\partial r}\left\{\frac{\partial u_2}{\partial r} + \frac{u_2}{r} + \frac{\partial v_2}{\partial z}\right\} \\ &\qquad = \frac{\partial u_2}{\partial t} + \rho_1\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial r} + v_1\frac{\partial u_1}{\partial z}, \\ &-\frac{\partial p_2}{\partial z} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v_2}{\partial r^2} + \frac{1}{r}\frac{\partial v_2}{\partial r} + \frac{\partial^2 v_2}{\partial z^2}\right) + \frac{1}{3\,\mathrm{Re}}\frac{\partial}{\partial z}\left\{\frac{\partial u_2}{\partial r} + \frac{u_2}{r} + \frac{\partial v_2}{\partial z}\right\} \\ &\qquad = \frac{\partial v_2}{\partial t} + \rho_1\frac{\partial v_1}{\partial t} + u_1\frac{\partial v_1}{\partial r} + v_1\frac{\partial v_1}{\partial z}, \\ &\frac{\partial \rho_2}{\partial t} + \frac{\partial u_2}{\partial r} + \frac{u_2}{r} + \frac{\partial v_2}{\partial z} + \rho_1\left(\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{\partial v_1}{\partial z}\right) + u_1\frac{\partial \rho_1}{\partial r} + v_1\frac{\partial \rho_1}{\partial z} = 0, \end{aligned} \tag{3.4}$$

$$\rho_2 = \chi p_2 + \tfrac{1}{2}\chi^2 p_1^2;$$

etc. We represent the boundary conditions (2.15) by a Taylor expansion around $r = 1$, *i.e.*

$$\begin{aligned} &v_r(1,z,t) + \eta(z,t)\partial r^2 \frac{\partial v_r}{\partial r}(1,z,t) + \frac{\eta^2(z,t)}{2}\frac{\partial^2 v_r}{\partial r^2}(1,z,t) + \cdots = \varepsilon\alpha \sin\alpha(z-t), \\ &v_z(1,z,t) + \eta(z,t)\frac{\partial v_z}{\partial r}(1,z,t) + \frac{\eta^2(z,t)}{2}\frac{\partial^2 v_z}{\partial r^2}(1,z,t) + \cdots = 0. \end{aligned} \tag{3.5}$$

Substituting expansions (3.1) in (3.5), and expressing cos and sin in exponential powers, we obtain the following sequence of boundary conditions

$$\begin{aligned} &u_1(1,z,t) = -\frac{\alpha i}{2}\left(\mathrm{e}^{i\alpha(z-t)} - \mathrm{e}^{-i\alpha(z-t)}\right), \\ &u_2(1,z,t) + \tfrac{1}{2}\left(\mathrm{e}^{i\alpha(z-t)} + \mathrm{e}^{-i\alpha(z-t)}\right)\frac{\partial u_1}{\partial r}(1,z,t) = 0, \\ &v_1(1,z,t) = 0, \\ &v_2(1,z,t) + \tfrac{1}{2}\left(\mathrm{e}^{i\alpha(z-t)} + \mathrm{e}^{-i\alpha(z-t)}\right)\frac{\partial v_1}{\partial r}(1,z,t) = 0. \end{aligned} \tag{3.6}$$

Finally, we represent the flow rate Q by a Taylor expansion around $r = 1$, and obtain by use of boundary condition (3.6)[3]

$$Q(z,t) = 2\pi\left[\varepsilon\int_0^1 v_1(r,z,t)r\,\mathrm{d}r + \varepsilon^2\int_0^1 v_2(r,z,t)r\,\mathrm{d}r + O(\varepsilon^3)\right]. \tag{3.7}$$

Examination of Equations (3.3)–(3.6) shows that a solution can be chosen in the form

$$\begin{aligned} &u_1(r,z,t) = U_1(r)\,\mathrm{e}^{i\alpha(z-t)} + \overline{U_1}(r)\,\mathrm{e}^{-i\alpha(z-t)}, \\ &v_1(r,z,t) = V_1(r)\,\mathrm{e}^{i\alpha(z-t)} + \overline{V_1}(r)\,\mathrm{e}^{-i\alpha(z-t)}, \\ &p_1(r,z,t) = P_1(r)\,\mathrm{e}^{i\alpha(z-t)} + \overline{P_1}(r)\,\mathrm{e}^{-i\alpha(z-t)}, \\ &\rho_1(r,z,t) = \chi P_1(r)\,\mathrm{e}^{i\alpha(z-t)} + \chi\overline{P_1}(r)\,\mathrm{e}^{-i\alpha(z-t)}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &u_2(r,z,t) = U_{20}(r) + U_2(r)\,\mathrm{e}^{2i\alpha(z-t)} + \overline{U_2}(r)\,\mathrm{e}^{-2i\alpha(z-t)}, \\ &v_2(r,z,t) = V_{20}(r) + V_2(r)\,\mathrm{e}^{2i\alpha(z-t)} + \overline{V_2}(r)\,\mathrm{e}^{-2i\alpha(z-t)}, \\ &p_2(r,z,t) = P_{20}(r) + P_2(r)\,\mathrm{e}^{2i\alpha(z-t)} + \overline{P_2}(r)\,\mathrm{e}^{-2i\alpha(z-t)}, \\ &\rho_2(r,z,t) = D_{20}(r) + D_2(r)\,\mathrm{e}^{2i\alpha(z-t)} + \overline{D_2}(r)\,\mathrm{e}^{-2i\alpha(z-t)}. \end{aligned} \tag{3.9}$$

Here, the overbar denotes the complex conjugate.

Substituting (3.8) in (3.3) and (3.6), we obtain the following set of equations:

$$\begin{aligned}
&-P_1' + \frac{1}{\mathrm{Re}}\left(U_1'' + \frac{U_1'}{r} - \frac{U_1}{r^2} - \alpha^2 U_1\right) + \frac{1}{3\,\mathrm{Re}}\frac{\mathrm{d}}{\mathrm{d}r}\left\{U_1' + \frac{U_1}{r} + i\alpha V_1\right\} = -i\alpha U_1,\\
&-i\alpha P_1 + \frac{1}{\mathrm{Re}}\left(V_1'' + \frac{V_1'}{r} - \alpha^2 V_1\right) + \frac{i\alpha}{3\,\mathrm{Re}}\left(U_1' + \frac{U_1}{r} + i\alpha V_1\right) = -i\alpha V_1,\\
&U_1' + \frac{U_1}{r} + i\alpha V_1 = i\alpha\chi P_1,\\
&U_1(1) = -\frac{i\alpha}{2}, \qquad V_1(1) = 0.
\end{aligned} \tag{3.10}$$

The set of equations for $\overline{U_1}$, $\overline{V_1}$ and $\overline{P_1}$ is conjugate to (3.10), so that it need not be written out explicitly. Firstly, the system of equations (3.10) is solved. Substitution of $(3.10)^3$ in $(3.10)^{1,2}$ yields

$$-\gamma P_1' + \left(U_1'' + \frac{U_1'}{r} - \frac{U_1}{r^2} - \beta^2 U_1\right) = 0, \tag{3.11}$$

$$-\gamma P_1 - \frac{i}{\alpha}\left(V_1'' + \frac{V_1'}{r} - \beta^2 V_1\right) = 0,$$

where the (complex) parameters γ and β are given by

$$\gamma = \mathrm{Re} - \frac{i\alpha\chi}{3}, \qquad \beta^2 = \alpha^2 - i\alpha\,\mathrm{Re}. \tag{3.12}$$

With V_1 eliminated by means of $(3.10)^3$, we rewrite $(3.11)^2$ as

$$\begin{aligned}
&-\frac{i\chi}{\alpha}\left(P_1'' + \frac{P_1'}{r} - \left(\beta^2 + \frac{\alpha\gamma i}{\chi}\right)P_1\right)\\
&+\frac{1}{\alpha^2}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r}\right)\left\{U_1'' + \frac{U_1'}{r} - \frac{U_1}{r^2} - \beta^2 U_1\right\} = 0.
\end{aligned} \tag{3.13}$$

Differentiating (3.13) with respect to r and using $(3.11)^1$ to express the derivatives of P_1 in terms of U_1 and its derivatives, we are led to the following equation

$$\begin{aligned}
&-\frac{i\chi}{\alpha\gamma}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2} - \left(\beta^2 + \frac{\alpha\gamma i}{\chi}\right)\right)\left\{U_1'' + \frac{U_1'}{r} - \frac{U_1}{r^2} - \beta^2 U_1\right\}\\
&+\frac{1}{\alpha^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2}\right)\left\{U_1'' + \frac{U_1'}{r} - \frac{U_1}{r^2} - \beta^2 U_1\right\} = 0.
\end{aligned} \tag{3.14}$$

This equation is rewritten, after multiplication by α^2, as

$$B\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2} - \nu^2\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2} - \beta^2\right)U_1 = 0, \tag{3.15}$$

where the (complex) parameters B and ν are given by

$$B = 1 - \frac{i\alpha\chi}{\gamma}, \qquad \nu^2 = \beta^2 - \frac{\beta^2 - \alpha^2}{B}. \tag{3.16}$$

By use of (3.12), expression (3.16) for ν is written in terms of α, Re and χ as

$$\nu^2 = \alpha^2 \frac{(1-\chi)\,\mathrm{Re} - \frac{4}{3} i\alpha\chi}{\mathrm{Re} - \frac{4}{3} i\alpha\chi}. \tag{3.17}$$

Notice that $\nu = \alpha$ for $\chi = 0$, *i.e.* for incompressible liquids. The solution for U_1, with $U_1(r)$ regular in $r = 0$, is given by

$$U_1(r) = C_1 I_1(\nu r) + C_2 I_1(\beta r), \tag{3.18}$$

where I_1 is the modified Bessel function of the first kind of order 1, and C_1 and C_2 are (complex) constants. Substitution of (3.18) in $(3.11)^1$ yields

$$\gamma P_1'(r) = C_1(\nu^2 - \beta^2) I_1(\nu r). \tag{3.19}$$

By use of the property $I_1(z) = I_0'(z)$, where I_0 is the modified Bessel function of the first kind of order 0, it is easily seen that

$$P_1(r) = C_3 + C_1 \frac{\nu^2 - \beta^2}{\nu\gamma} I_0(\nu r), \tag{3.20}$$

where C_3 is a (complex) constant. Substituting (3.18) and (3.20) in $(3.10)^3$, and using the property $I_1'(z) + I_1(z)/z = I_0(z)$, we obtain

$$V_1(r) = \frac{i\alpha C_1}{\nu} I_0(\nu r) + \frac{i\beta C_2}{\alpha} I_0(\beta r) + \chi C_3, \tag{3.21}$$

where (3.12) and (3.17) have been used for β and ν. Finally, by putting (3.20) into $(3.11)^2$ we deduce that

$$C_3 = 0. \tag{3.22}$$

The boundary conditions $(3.10)^{4,5}$ then require

$$C_1 = \frac{\alpha\beta\nu i I_0(\beta)}{2[\alpha^2 I_0(\nu) I_1(\beta) - \beta\nu I_0(\beta) I_1(\nu)]}, \; C_2 = \frac{-\alpha^3 i I_0(\nu)}{2[\alpha^2 I_0(\nu) I_1(\beta) - \beta\nu I_0(\beta) I_1(\nu)]}. \tag{3.23}$$

Thus, the first-order approximation of the velocity and pressure is described by Equations (3.18) and (3.20)–(3.23).

On inserting (3.9) into (3.4) and (3.6), we obtain the following set of equations:

$$
\begin{aligned}
&-P_{20}' + \frac{1}{\mathrm{Re}}\left(U_{20}'' + \frac{U_{20}'}{r} - \frac{U_{20}}{r^2}\right) + \frac{1}{3\,\mathrm{Re}}\frac{\mathrm{d}}{\mathrm{d}r}\left\{U_{20}' + \frac{U_{20}}{r}\right\} \\
&\quad = i\alpha\chi P_1\overline{U_1} - i\alpha\chi\overline{P_1}U_1 + U_1\overline{U_1'} + \overline{U_1}U_1' - i\alpha V_1\overline{U_1} + i\alpha\overline{V_1}U_1, \\
&\frac{1}{\mathrm{Re}}\left(V_{20}'' + \frac{V_{20}'}{r}\right) = i\alpha\chi\overline{V_1}P_1 - i\alpha\chi\overline{P_1}V_1 + U_1\overline{V_1'} + \overline{U_1}V_1', \\
&U_{20}' + \frac{U_{20}}{r} = -\chi\left(P_1\overline{U_1'} + \overline{P_1}U_1' + \frac{P_1\overline{U_1}}{r} + \frac{\overline{P_1}U_1}{r} + U_1\overline{P_1'} + \overline{U_1}P_1'\right), \\
&D_{20} = \chi P_{20} + \chi^2\overline{P_1}P_1, \\
&U_{20}(1) + \tfrac{1}{2}\overline{U_1'}(1) + \tfrac{1}{2}U_1'(1) = 0, \qquad V_{20}(1) + \tfrac{1}{2}\overline{V_1'}(1) + \tfrac{1}{2}V_1'(1) = 0.
\end{aligned}
\tag{3.24}
$$

It will be seen later that, as far as the net flow is considered, only the functions U_{20}, V_{20}, P_{20} and D_{20} participate in the solution, as long as terms up to $O(\varepsilon^2)$ are retained. Thus, the functions U_2, V_2, P_2 and D_2 do not contribute to the net flow, and therefore, we shall not write down the equations that these functions satisfy, nor solve them. We continue with the solutions for P_{20}, V_{20} and U_{20}. To that end, we rewrite the Equations $(3.24)^{1,2,3}$ as

$$
\begin{aligned}
&-P_{20}' + \frac{4}{3\,\mathrm{Re}}\left(U_{20}'' + \frac{U_{20}'}{r} - \frac{U_{20}}{r^2}\right) = F, \\
&V_{20}'' + \frac{V_{20}'}{r} = \mathrm{Re}\,G, \\
&U_{20}' + \frac{U_{20}}{r} = -\chi H,
\end{aligned}
\tag{3.25}
$$

where the functions F, G and H, corresponding to the right-hand sides of $(3.24)^{1,2,3}$, are given by

$$
\begin{aligned}
&F = i\alpha\chi\left(P_1\overline{U_1} - \overline{P_1}U_1\right) + U_1\overline{U_1'} + \overline{U_1}U_1' + i\alpha\left(\overline{V_1}U_1 - V_1\overline{U_1}\right), \\
&G = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r(V_1\overline{U_1} + \overline{V_1}U_1)\right\}, \\
&H = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r(P_1\overline{U_1} + \overline{P_1}U_1)\right\}.
\end{aligned}
\tag{3.26}
$$

Here, Equation $(3.10)^3$ is used to eliminate P_1 in function G. With H given by $(3.26)^3$ it is easily seen that the solution for U_{20} is given by

$$
U_{20}(r) = \frac{D_1}{r} - \chi\left(P_1(r)\overline{U_1}(r) + \overline{P_1}(r)U_1(r)\right), \tag{3.27}
$$

where the (complex) constant D_1 follows from the boundary condition $(3.24)^5$. Elimination of U_1' by means of $(3.10)^3$ and by use of $(3.10)^{4,5}$ leads to

$$D_1 = 0. \tag{3.28}$$

Notice that $U_{20} = 0$ for $\chi = 0$, *i.e.* for incompressible liquids. With G given by $(3.26)^2$, it is easily seen that the solution for V_{20}, with $V_{20}(r)$ regular in $r = 0$, is given by

$$V_{20}(r) = D_2 - \mathrm{Re} \int_r^1 [V_1(y)\overline{U_1}(y) + \overline{V_1}(y)U_1(y)]\,\mathrm{d}y. \tag{3.29}$$

Here the (complex) constant $D_2 = V_{20}(1)$, where $V_{20}(1)$ is given by $(3.24)^6$, equals

$$D_2 = -\frac{i\alpha C_1}{2} I_1(\nu) - \frac{i\beta^2 C_2}{2\alpha} I_1(\beta) + \frac{i\alpha \overline{C_1}}{2} I_1(\overline{\nu}) + \frac{i\overline{\beta}^2\,\overline{C_2}}{2\alpha} I_1(\overline{\beta}). \tag{3.30}$$

Finally, it is easily seen that the solution for P_{20} is given by

$$P_{20}(r) = D_3 - \frac{4\chi}{3\,\mathrm{Re}} H(r) - \int_0^r F(y)\,\mathrm{d}y. \tag{3.31}$$

Here, the (complex) constant $D_3 = P_{20}(0) + 4\chi H(0)/3\,\mathrm{Re}$ follows from a given pressure $P_{20}(0)$ on the centerline of the pore. Elaboration of $H(0)$ yields

$$D_3 = P_{20}(0) + \frac{4\chi}{3\,\mathrm{Re}} \left[\frac{\nu^2 - \beta^2}{\nu\gamma} C_1 \overline{(\nu C_1 + \beta C_2)} + \overline{\frac{\nu^2 - \beta^2}{\nu\gamma} C_1} (\nu C_1 + \beta C_2) \right]. \tag{3.32}$$

Next, the net flow is considered which is the flow averaged over one period of time. To that end, we introduce the average of a variable g over one period $2\pi/\alpha$ of time t as

$$\langle g \rangle = \frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} g(r, z, t)\,\mathrm{d}t. \tag{3.33}$$

Consequently, the net axial velocity $\langle v_z \rangle$ reads

$$\langle v_z \rangle = \varepsilon^2 V_{20}(r), \tag{3.34}$$

under neglect of $O(\varepsilon^3)$-terms, while the net flow rate $\langle Q \rangle$ is given by

$$\langle Q \rangle = 2\pi\varepsilon^2 \int_0^1 V_{20}(r) r\,\mathrm{d}r, \tag{3.35}$$

under neglect of $O(\varepsilon^3)$-terms. Thus, the travelling wave induces a net flow of the liquid, of which the (dimensionless) rate is expressed by (3.35). Hence, the net flow is an effect of order ε^2.

4. Numerical results

To study the behavior of the net flow, numerical calculations for several values of ε, α, Re and χ are carried out. Firstly, we concentrate on the solution of the dimensionless problem as described in the previous section. The condition for the solution presented by the formal expansions (3.1) to be accurate, is that $\varepsilon\alpha^2\,\mathrm{Re} \ll 1$ according to Takabatake [7].

We consider the net flow rate $\langle Q\rangle$ given by (3.35). After one integration by parts $\langle Q\rangle$ can be expressed as

$$\langle Q\rangle = \pi\varepsilon^2\left(D_2 - \mathrm{Re}\int_0^1 r^2[V_1(r)\overline{U_1}(r) + \overline{V_1}(r)U_1(r)]\,\mathrm{d}r\right), \tag{4.1}$$

where the solution (3.29) for V_{20} is used.

For incompressible liquids we compare the net flow rate $\langle Q\rangle$ to the net flow rate $\overline{q}$ given by Yin and Fung [6, formula (54)]. Elaboration of the results of [6] yields

$$\overline{q} = -\pi\varepsilon^2\left(\zeta + \frac{\alpha^2\,\mathrm{Re}^2}{8}\sum_{j=1}^{M}\frac{B_j}{j(j+1)(j+2)}\right), \tag{4.2}$$

where the constants ζ and B_j are given in Table 1 and 2 of [6]. Notice that the constant L_3 in [6] is set to zero. For instance, for $\varepsilon = 0{\cdot}15$, Re$=$ 100 and $\alpha = 0{\cdot}2$ we obtain $\langle Q\rangle = 0{\cdot}2709$ and $\overline{q} = 0{\cdot}2708$. More calculations disclose that for $\chi = 0$ our results correspond to the results of Yin and Fung [6].

In Figure 2 the net flow rate $\langle Q\rangle$ is plotted versus χ, for $\varepsilon = 0{\cdot}001$, $\alpha = 0{\cdot}001$, and $\mathrm{Re} = 1000$ (dashed-dotted curve), $\mathrm{Re} = 5000$ (dashed curve) and $\mathrm{Re} = 10{,}000$ (solid curve). We observe that the range of $\langle Q\rangle$ is approximately $0{\cdot}01$–$1{\cdot}26 \times 10^{-5}$ if $\mathrm{Re} = 1000$, $0{\cdot}15$–$1{\cdot}68 \times 10^{-5}$ if $\mathrm{Re} = 5000$ and $0{\cdot}49$–$2{\cdot}64 \times 10^{-5}$ if $\mathrm{Re} = 10{,}000$, for $0 \leqslant \chi \leqslant 1$. In particular, for $\chi = 0$ the range of $\langle Q\rangle$ is just $1{\cdot}24$–1.26×10^{-5} for the three values of Re considered, while for $\chi > 0$ the range becomes $0{\cdot}01$–$2{\cdot}64 \times 10^{-5}$. Hence, $\langle Q\rangle$ is nearly independent of Re, for $\chi = 0$. More precisely, the calculations disclose the result $\langle Q\rangle \approx 4\pi\varepsilon^2$, for $\alpha \ll 1$ and $\chi = 0$. For $\chi > 0$, however, $\langle Q\rangle$ strongly depends on Re. Furthermore, we observe that for $\mathrm{Re} = 1000$ the net flow rate $\langle Q\rangle$ decreases with increasing χ. For $\mathrm{Re} = 5000$ and 10,000, on the other hand, $\langle Q\rangle$ attains a maximum of $1{\cdot}7 \times 10^{-5}$ at $\chi = 0{\cdot}2$ and $2{\cdot}7 \times 10^{-5}$ at $\chi = 0.5$, respectively. Thus, the compressibility number χ has a significant influence on the net flow rate, and the Reynolds number Re plays a more significant role in the net flow of a compressible liquid than of an incompressible one.

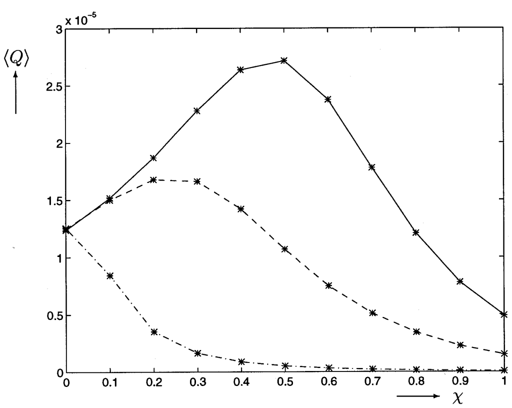

Figure 2. The (dimensionless) net flow rate $\langle Q\rangle$ versus χ, for $\varepsilon = 0{\cdot}001$, $\alpha = 0{\cdot}001$, and $\mathrm{Re} = 1000$ (dashed-dotted curve), $\mathrm{Re} = 5000$ (dashed curve) and $\mathrm{Re} = 10{,}000$ (solid curve).

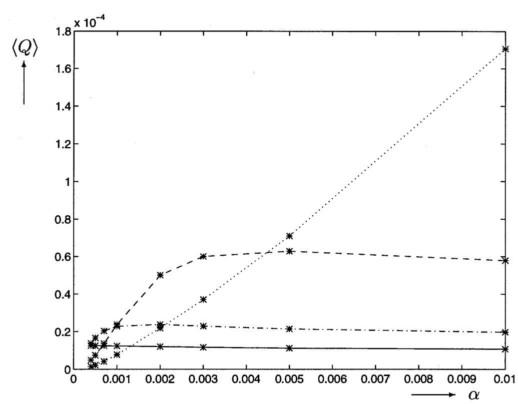

Figure 3. The (dimensionless) net flow rate $\langle Q\rangle$ versus α, for $\varepsilon = 0{\cdot}001$, $\mathrm{Re} = 10^4$, and $\chi = 0{\cdot}0$ (solid curve), $\chi = 0{\cdot}3$ (dashed-dotted curve), $\chi = 0{\cdot}6$ (dashed curve) and $\chi = 0{\cdot}9$ (dotted curve).

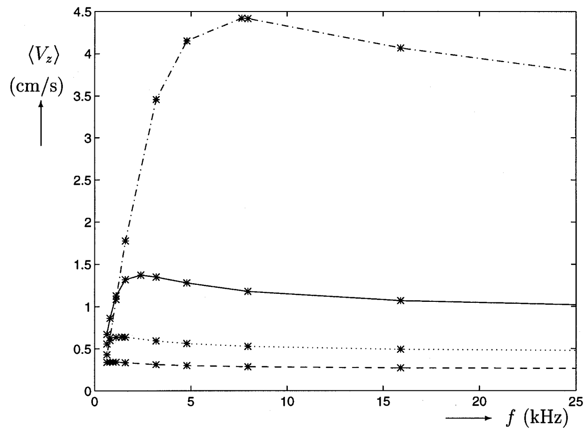

Figure 4. The mean net axial velocity $\langle V_z \rangle$ (in cm/s) versus the frequency f (in kHz), for $\kappa = 0{\cdot}5 \times 10^{-9}$ N^{-1}m^2, $\mu = 0{\cdot}5 \times 10^{-2}$ Nm^{-2}s, $\rho_0 = 10^3$ kg/m^3, $R = 10^{-4}$ m and $a = 10^{-7}$ m, for $c = 600$ ms^{-1} (dashed curve), $c = 800$ ms^{-1} (dotted curve), $c = 1000$ ms^{-1} (solid curve) and $c = 1200$ ms^{-1} (dashed-dotted curve).

In Figure 3 the net flow rate $\langle Q \rangle$ is plotted versus α, for $\varepsilon = 0{\cdot}001$, Re $= 10^4$, and $\chi = 0{\cdot}0$ (solid curve), $\chi = 0{\cdot}3$ (dashed-dotted curve), $\chi = 0{\cdot}6$ (dashed curve) and $\chi = 0{\cdot}9$ (dotted curve). We observe that the range of $\langle Q \rangle$ is approximately $1{\cdot}12$–$1{\cdot}25 \times 10^{-5}$ if $\chi = 0$, $0{\cdot}99$–$2{\cdot}29 \times 10^{-5}$ if $\chi = 0{\cdot}3$, $0{\cdot}29$–$6{\cdot}28 \times 10^{-5}$ if $\chi = 0{\cdot}6$ and $0{\cdot}09$–$17{\cdot}06 \times 10^{-5}$ if $\chi = 0{\cdot}9$, for $0{\cdot}0004 \leqslant \alpha \leqslant 0{\cdot}01$. Furthermore, if $\chi = 0{\cdot}3$ or $0{\cdot}6$, then $\langle Q \rangle$ attains a maximum for a certain value of α, and this maximum increases with increasing χ. For $\chi = 0$, on the other hand, $\langle Q \rangle$ is decreasing with increasing α. More precisely, $\langle Q \rangle$ is nearly independent of α, which again corresponds to the calculated result $\langle Q \rangle \approx 4\pi\varepsilon^2$, for $\alpha \ll 1$ and $\chi = 0$. For $\chi = 0{\cdot}9$, the net flow rate increases with increasing α.

Next, we return to the dimensional flow problem, as described in Section 2. The dimensional net flow rate $\langle Q \rangle$ is equal to the dimensionless net flow rate $\langle Q \rangle$ as given by (4.1) multiplied by the factor cR^2. In addition, we define the dimensional mean net axial velocity $\langle V_z \rangle$ as

$$\langle V_z \rangle = \frac{\langle Q \rangle}{\pi R^2}, \tag{4.3}$$

where $\langle Q \rangle$ is the dimensional net flow rate, and πR^2 denotes the mean cross-sectional area of the pore. The amplitude $\mathcal{A}_p$ of the dimensional pressure wave $p = \rho_0 c^2(\varepsilon p_1 + O(\varepsilon^2))$, where p_1 is given by (3.8), is defined as

$$\mathcal{A}_p(r) = 2\rho_0 c^2 \varepsilon \sqrt{[\mathfrak{Re}\{P_1(r)\}]^2 + [\mathfrak{Im}\{P_1(r)\}]^2}. \tag{4.4}$$

The maximum value of this amplitude is defined as the norm of p, that is $\|p\| := \max_{r\in[0,1]} \mathcal{A}_p(r)$. Calculations disclose that the distribution of $\mathcal{A}_p(r)$ is nearly uniform along the radius r of the pore, which has also been found by Ganiev *et al.* [4].

We consider the case in which the pore has radius $R = 10^{-4}$ m, and the wave has amplitude $a = 10^{-7}$ m. The properties of the liquid are given by $\rho_0 = 10^3$ kg/m^3, $\kappa = 0{\cdot}5 \times 10^{-9}$ N^{-1}m^2 and $\mu = 0{\cdot}5 \times 10^{-2}$ Nm^{-2}s. The frequency f of the wave is related to the wave speed c and the wavelength λ according to

$$c = \lambda f. \tag{4.5}$$

Table 1. The frequency $f_{\max}$ at which the maximum of $\langle V_z \rangle$ is attained, the maximum $\langle V_z \rangle_{\max}$ itself, the corresponding net flow rate $\langle Q \rangle_{\max}$ and norm $\|p\|$ of the pressure wave, for $R = 10^{-4}$ m, $a = 10^{-7}$ m, $\rho_0 = 10^3$ kg/m^3, $\mu = 0{\cdot}5 \times 10^{-2}$ Nm^{-2}s and $\kappa = 0{\cdot}5 \times 10^{-9}$ N^{-1}m^2.

c (m/s)	$f_{\max}$ (Hz)	$\langle V_z \rangle_{\max}$ (cm/s)	$\langle Q \rangle_{\max}$ (mm^3/s)	$\|p\|$ ($\times 10^5$ Nm^{-2})
600	960	0·34	0·11	14
700	1110	0·46	0·15	20
800	1400	0·64	0·20	29
900	1590	0·91	0·29	42
1000	2390	1·37	0·43	61
1100	3680	2·25	0·71	92
1200	7640	4·42	1·39	151

For various values of the wave speed c we calculate the mean net axial velocity $\langle V_z \rangle$ as a function of the frequency f. Thus, $\varepsilon = 0{\cdot}001$ is constant, $\mathrm{Re} = \rho_0 c R/\mu$ and $\chi = \kappa \rho_0 c^2$ increase with increasing c, while α increases with increasing f according to $\alpha = 2\pi R f/c = 6{\cdot}28 \times 10^{-4} f/c$ for the various values of c considered. In Figure 4 the mean net axial velocity $\langle V_z \rangle$ (in cm/s) is plotted versus f (in kHz), for $c = 600\,\mathrm{ms}^{-1}$ (dashed curve), $c = 800\,\mathrm{ms}^{-1}$ (dotted curve), $c = 1000\,\mathrm{ms}^{-1}$ (solid curve) and $c = 1200\,\mathrm{ms}^{-1}$ (dashed-dotted curve). We observe that the range of $\langle V_z \rangle$ is approximately $0{\cdot}26 - 0.34$ cm/s if $c = 600\,\mathrm{ms}^{-1}$, 0·47–0.64 cm/s if $c = 800\ \mathrm{ms}^{-1}$, 0·67–1·37 cm/s if $c = 1000\,\mathrm{ms}^{-1}$, and 0·43–4·42 cm/s if $c = 1200\,\mathrm{ms}^{-1}$, for the values of f considered. Furthermore, $\langle V_z \rangle$ attains a maximum for a certain value of f. In Table 1 the frequency $f_{\max}$ at which the maximum of $\langle V_z \rangle$ is attained, the maximum $\langle V_z \rangle_{\max}$ itself, the corresponding net flow rate $\langle Q \rangle_{\max}$ and norm $\|p\|$ of the pressure wave are given. We observe that $f_{\max}$ as well as the corresponding mean net axial velocity $\langle V_z \rangle_{\max}$ increases with increasing wave speed c.

5. Discussion

In order to analyze the influence of ultrasonic radiation on the flow of oil through a reservoir, we assume that the deformation of the pore walls is caused by excess pressure waves generated by an acoustic source. The source is placed in the well at a distance h from the oil reservoir, where the oil has compressibility κ, density ρ_0 at reference pressure p_0, and viscosity μ. By radiation from a simple source in radial direction the excess pressure $p - p_0$ generated in the oil reads, in the linear theory of sound and under neglect of gravity and viscosity,

$$p - p_0 = \frac{1}{h}\sqrt{\frac{\rho_0 c_0 P}{2\pi}}, \tag{5.1}$$

provided that $h \gg \lambda/2\pi$; see Lighthill [8, Section 1.12]. Here, P is the average power output generated at the acoustic source, and $c_0 = 1/\sqrt{\kappa \rho_0}$ is the speed of sound in the oil. Since the porous medium is elastic, the displacement a of the pore wall due to the excess pressure is given by

$$a = \frac{(p - p_0)R}{2G}, \tag{5.2}$$

under the assumption that the pores are sufficiently far away from each other; see Jaeger and Cook [9, Section 5.11]. Here, G is the shear modulus of the porous medium. By a combination of (5.1) and (5.2), an estimation for the ratio $\varepsilon = a/R$ is given by

$$\varepsilon = \frac{1}{2hG}\sqrt{\frac{\rho_0 c_0 P}{2\pi}}. \tag{5.3}$$

Notice that according to (5.3) the net flow rate $\langle Q \rangle$ is proportional to the power output P, since the net flow is an effect of order ε^2. Furthermore, $\langle Q \rangle$ is proportional to $1/h^2$, so that the effects of ultrasonic radiation in radial direction are expected to be local, *i.e.* mainly in the near-wellbore region. If the radiation of the source is ducted *one-dimensionally*, on the other hand, the acoustic excess pressure $p - p_0$ is independent of the distance h, and the power output is much more efficiently generated; *cf.* Lighthill [8, Section 1.4].

The wave speed c on the pore wall depends on the compressibility κ of the liquid as well as on the distensibility D of the pore. Lighthill [8, pp. 93] defines the wave speed as

$$c = \frac{1}{\sqrt{\rho_0(\kappa + D)}}, \tag{5.4}$$

where the distensibility is given by

$$D = \left(\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}p}\right)_{p=p_0}. \tag{5.5}$$

Here, A denotes the cross-sectional area of the pore. With $A = \pi(R + a)^2$ and a given by (5.2), the cross-sectional area can be expressed in terms of the excess pressure as

$$A = \pi R^2 \left(1 + \frac{p - p_0}{2G}\right)^2 \tag{5.6}$$

Substitution of (5.6) in (5.5) gives

$$D = \frac{1}{G}. \tag{5.7}$$

Notice that, since $D > 0$ and $c_0 = 1/\sqrt{\kappa\rho_0}$, the wave speed c according to (5.4) satisfies $c/c_0 < 1$, so that $\chi = c^2/c_0^2 = \kappa/(\kappa + D) < 1$. Thus, by means of (5.3) the wave amplitude is related to the power output generated at the acoustic source, while by means of (5.4) and (5.7) the wave speed on the pore wall is expressed in terms of the shear modulus of the porous medium.

As an example, let the power output generated at the acoustic source be $P = 10$ kW, while the source is placed at a distance $h = 0{\cdot}05$ m from the reservoir. For the properties of the oil we take $\rho_0 = 800$ kg/m^3 and $\kappa = 0{\cdot}7 \times 10^{-9}\,\mathrm{N}^{-1}\mathrm{m}^2$, so that $c_0 = 1336$ m/s.

By substitution of the values in (5.1) we obtain an excess pressure generated by the acoustic source of $p - p_0 = 8{\cdot}2 \times 10^5\,\mathrm{Nm}^{-2}$. For a sandstone the shear modulus is approximately $G = 0{\cdot}5 \times 10^9\,\mathrm{Nm}^{-2}$. As a result, the ratio of the wave amplitude a to the pore radius R is given by $\varepsilon = 0{\cdot}82 \times 10^{-3}$.

By substitution of the values in (5.4) the wave speed of the travelling wave is $c = 680\,\mathrm{m/s}$, where $D = 2 * 10^{-9}\,\mathrm{N}^{-1}\mathrm{m}^2$ is used. As a result, the dimensionless parameter $\chi = c^2/c_0^2$ becomes $\chi = 0{\cdot}26$. Let the frequency of the acoustic source, and thus of the travelling wave, be equal to $f = 20\,\mathrm{kHz}$, then the wave length is $\lambda = c/f = 0{\cdot}034\,\mathrm{m}$. Finally, let the pores of the reservoir have a radius of $R = 0{\cdot}5 \times 10^{-4}$ m and the oil have a viscosity of $\mu = 5 \times 10^{-3}\,\mathrm{Nm}^{-2}\mathrm{s}$. Then the dimensionless parameters $\alpha = 2\pi R/\lambda$ and $\mathrm{Re} = \rho_0 c R/\mu$ become $\alpha = 9{\cdot}23 \times 10^{-3}$ and $\mathrm{Re} = 5443$.

Calculations for the specific values disclose a net flow rate of $\langle Q \rangle = 0{\cdot}223 \times 10^{-10}\,\mathrm{m}^3/\mathrm{s}$. As already observed, the net flow rate induced by the travelling wave is small. However, this flow rate is of the same order of magnitude as the flow rate Q_{Pois} of the Poiseuille flow through the pore induced by the pressure gradient $\mathrm{d}p/\mathrm{d}z$ in the reservoir, which is given by

$$Q_{\mathrm{Pois}} = \frac{\mathrm{d}p}{\mathrm{d}z}\frac{\pi R^4}{8\mu}. \tag{5.8}$$

Therefore, in an oil reservoir a typical value of the pressure gradient is $(\mathrm{d}p/\mathrm{d}z) = 10^5\,\mathrm{N/m}^3$. By substitution of the specific values in (5.8), we arrive at $Q_{\mathrm{Pois}} = 0{\cdot}491 \times 10^{-10}\mathrm{m}^3/\mathrm{s}$, so that $\langle Q \rangle \approx 0{\cdot}45 \times Q_{\mathrm{Pois}}$ in our example. Thus, although the net flow induced by travelling waves is an effect which is proportional to ε^2, this effect cannot be neglected in the flow through an oil reservoir. Notice that the flow rate Q_{Pois} strongly decreases with decreasing R, while the net flow rate $\langle Q \rangle$ induced by a travelling wave is not that sensitive to the pore radius. Furthermore, in order to analyse the net flow induced if a Poiseuille flow is already present when the acoustic source is off, the zeroth-order velocity V_0 according to (3.2) must be included in the solution.

References

1. W.-I Chen, *Influence of Ultrasonic Energy upon the Rate of Flow of Liquids through Porous Media*. Ph.D. thesis, West Virginia University (1969) 130 pp.
2. N. V. Cherskiy, V. P. Tsarev, V. M. Konovalov and O. L. Kuznetsov, The effect of ultrasound on permeability of rocks to water. *Trans. (Doklady) USSR Acad. Sci., Earth Sci. Sect.* 232 (1977) 201–204.
3. R. D. Duhon, *An Investigation of the Effect of Ultrasonic Energy on the Flow of Fluids in Porous Media*. Ph.D. thesis, University of Oklahoma (1964) 162 pp.
4. R. F. Ganiev, L. E. Ukrainskii and K. V. Frolov, Wave mechanism for the acceleration of a liquid flowing in capillaries and porous media. *Sov. Phys. Dokl.* 34 (1989) 519–521.
5. A. H. Shapiro, M. Y. Jaffrin and S. L. Weinberg, Peristaltic pumping with long wavelengths at low Reynolds number. *J. Fluid Mech.* 37 (1969) 799–825.
6. F. Yin and Y. C. Fung, Peristaltic waves in circular cylindrical tubes. *J. Appl. Mech.* 36 (1969) 579–587.
7. S. Takabatake, K. Ayukawa and A. Mori, Peristaltic pumping in circular cylindrical tubes: a numerical study of fluid transport and its efficiency. *J. Fluid Mech.* 193 (1988) 267–283.
8. J. Lighthill, *Waves in fluids*. Cambridge: Cambridge University Press (1978) 504 pp.
9. J. C. Jaeger and N. G. W. Cook, *Fundamentals of Rock Mechanics*. London: Chapman and Hall (1979) 593 pp.